STEADY-STATE TEMPERATURE FIELD OF A DISC WITH CONVECTIVE HEAT TRANSFER AT ITS SURFACE

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The mixed problem with axial symmetry is solved approximately, with boundary conditions of the third kind specified at the disc surface.

For the solution of many problems in the theory of heat conduction it is necessary to determine the steady-state temperature field of a disc in an infinitely large homogeneous medium under conditions of convective heat transfer between them. Such a problem is equivalent to the problem of a half-space with a circle of unit radius on its boundary where conditions of the third kind are satisfied and the remainder of the boundary thermally insulated, this problem to be solved by integrating the Laplace equation

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}=0 \tag{1}$$

with the following boundary conditions (defined in dimensionless form):

$$T-k\frac{\partial T}{\partial z}=1,\quad r\leqslant 1,\ z=0; \tag{2}$$

$$\frac{\partial T}{\partial z}=0,\quad r>1,\ z=0, \tag{3}$$

where $k = 1/Bi > 0$.

The well known solution to this problem [1] involves the necessity of evaluating the Fredholm integral equation of the second kind with respect to some auxiliary function, which considerably limits the feasibility of even a numerical analysis. In view of this, we will solve the problem by the method of characteristic surfaces [2], by which it can be reduced to the mathematical model shown in Fig. 1b.

In order to find the temperature distribution $T(r, z)$ within the given region, we will consider the auxiliary problems of determining the functions $T_1(r, z)$ and $T_2(r, z)$, which are harmonic in the respective subregions Ω_1: $\{0 < r < \infty,\ z > 0\}$ and Ω_2: $\{0 < r < 1,\ -k < z < 0\}$, with the following additional conditions

$$\left.\frac{\partial T_1}{\partial z}\right|_{\substack{z=0\\ r<1}}=\left.\frac{\partial T_2}{\partial z}\right|_{\substack{z=0\\ r<1}},\qquad T_1\Big|_{\substack{z=0\\ r<1}}=T_2\Big|_{\substack{z=0\\ r<1}}. \tag{4}$$

Assuming that the distribution of the normal derivative at the interface between subregions Ω_1 and Ω_2 (i.e., in the plane $z = 0,\ r \leq 1$) is equal to some as yet unknown function

$$\left.\frac{\partial T}{\partial z}\right|_{\substack{z=0\\ r<1}}=f(r),$$

we will write the boundary conditions for the auxiliary functions $T_1(r, z)$ and $T_2(r, z)$ as

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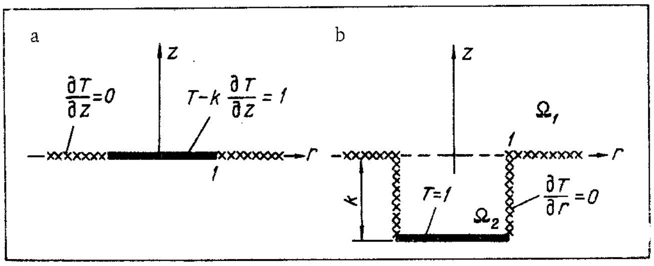

Fig. 1. Schematic diagram for specifying the boundary conditions: (a) prototype system, (b) auxiliary mathematical model.

$$\frac{\partial T_1}{\partial z} = \begin{cases} f(r) & r\leqslant 1, \quad z=0; \\ 0 & r>1, \quad z=0, \end{cases} \tag{7}$$

$$\left.\begin{aligned} &T_2 = 1 \qquad r\leqslant 1,\ z=-k; \\ &\frac{\partial T_2}{\partial r}=0 \qquad r=1, \quad -k<z<0; \\ &\frac{\partial T_2}{\partial z}=f(r) \quad r\leqslant 1, \quad z=0. \end{aligned}\right\} \tag{8}$$

The expression for T_1 is, as can be easily verified,

$$T_1 = \int_0^\infty A(\lambda)\lambda\exp(-\lambda z) J_0(\lambda r)\, d\lambda, \tag{9}$$

with

$$A(\lambda) = -\frac{1}{\lambda}\int_0^1 f(\rho) J_0(\lambda\rho)\rho d\rho$$

and

$$T_1 = -\int_0^1 f(\rho)\rho d\rho \int_0^\infty \exp(-\lambda z) J_0(\lambda\rho) J_0(\lambda r)\, d\lambda. \tag{10}$$

The solution in subregion Ω_2 is written in the form

$$T_2 = 1 + 2(z+k)\int_0^1 f(\rho)\rho d\rho + 2\sum_{n=1}^\infty \frac{\operatorname{sh}\alpha_n(z+k)}{\alpha_n \operatorname{ch}\alpha_n k}\cdot\frac{J_0(\alpha_n r)}{J_0^2(\alpha_n)}\int_0^1 f(\rho) J_0(\lambda\rho)\rho d\rho, \tag{11}$$

with α_n denoting the roots of the equation $J_1(\alpha_n) = 0$.

Taking into account the continuity of function $T(r, z)$ at $z = 0$, $r \le 1$ (condition (5)), we find from (10) and (11)

$$\int_0^1 f(\rho)\rho d\rho \int_0^\infty J_0(\lambda r) J_0(\lambda\rho)\, d\lambda + 2\sum_{n=1}^\infty \frac{\operatorname{th}\alpha_n k J_0(\alpha_n r)}{\alpha_n J_0^2(\alpha_n)}$$

$$\times \int_0^1 f(\rho) J_0(\alpha_n\rho)\rho d\rho + 2k\int_0^1 f(\rho)\rho d\rho = -1. \tag{12}$$

Considering also that [3]

$$\int_0^\infty J_0(\lambda\rho) J_0(\lambda r)\, d\lambda = \frac{1}{r} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\rho^2}{r^2}\right)$$

and

$$F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt,$$

we reduce (12) to

$$\frac{2}{\pi}\int\limits_0^1 f(\rho)\rho d\rho \int\limits_0^\rho \frac{dt}{\sqrt{r^2-t^2}\sqrt{\rho^2-t^2}} = -g(r), \tag{12'}$$

where

$$g(r) = 1 + 2k\int\limits_0^1 f(\rho)\rho d\rho + 2\sum_{n=1}^{\infty}\frac{\operatorname{th}\alpha_n k J_0(\alpha_n r)}{\alpha_n J_0^2(\alpha_n)}\int\limits_0^1 f(\rho) J_0(\alpha_n\rho)\rho d\rho.$$

Changing the order of integration in (12') from the left will yield the equation

$$\frac{2}{\pi}\int\limits_0^1 \frac{dt}{\sqrt{r^2-t^2}}\int\limits_t^1 \frac{f(\rho)\rho d\rho}{\sqrt{\rho^2-t^2}} = -g(r). \tag{13}$$

This equation can be solved by a subsequent application of the inversion formula for the Abel integral equation (see, e.g., [4]):

$$\rho f(\rho) = -\frac{2}{\pi}\cdot\frac{d}{d\rho}\int\limits_\rho^1 \frac{tdt}{\sqrt{t^2-\rho^2}}\cdot\frac{d}{dt}\int\limits_0^t \frac{rg(r)\,dr}{\sqrt{t^2-r^2}}. \tag{14}$$

Multiplying now both sides of (14) by $J_0(\alpha_m\rho)$ ($m = 0, 1, 2, \ldots$ and $\alpha_0 = 0$ when $m = 0$) and integrating with respect to ρ from 0 to 1, we obtain, after the necessary mathematical operations [3], the following infinite system of algebraic equations for the constants $\beta_0 = \int\limits_0^1 f(\rho)\rho d\rho$ and $\beta_n = \int\limits_0^1 f(\rho)J_0(\alpha_n\rho)\rho d\rho$.

$$\left.\begin{aligned}
\beta_0 &= \frac{1}{1+\frac{4k}{\pi}}\left\{-\frac{2}{\pi}-\frac{8}{\pi}\sum_{n=1}^{\infty}\beta_n\frac{\operatorname{th}\alpha_n k\sin\alpha_n}{\alpha_n^2 J_0^2(\alpha_n)}\right\},\\
\beta_m &= -\frac{2}{\pi}\cdot\frac{\sin\alpha_m}{\alpha_m}(1+2k\beta_0)-\frac{4}{\pi}\sum_{n=1}^{\infty}\beta_n\frac{\operatorname{th}\alpha_n k}{\alpha_n J_0^2(\alpha_n)}\\
&\times\left[\frac{\sin(\alpha_n+\alpha_m)}{\alpha_n+\alpha_m}+\frac{\sin(\alpha_n-\alpha_m)}{\alpha_n-\alpha_m}\right]\quad(m = 1,\ 2,\ 3,\ \ldots).
\end{aligned}\right\} \tag{15}$$

For the purpose of further analysis, it will be more convenient to transform system (15) to

$$\left.\begin{aligned}
\gamma_0 &= -\frac{2}{\pi}-2\sum_{n=1}^{\infty}\gamma_n\frac{\operatorname{th}\alpha_n k}{\operatorname{th}\alpha_n k+\frac{\pi}{4}\alpha_n J_0^2(\alpha_n)}\cdot\frac{\sin\alpha_n}{\alpha_n},\\
\gamma_m &= -\frac{2}{\pi}\cdot\frac{\sin\alpha_m}{\alpha_m}\left[1+2k\left(k+\frac{\pi}{4}\right)^{-1}\gamma_0\right]\\
&-\sum_{n=1}^{\infty}\gamma_n\frac{\operatorname{th}\alpha_n k}{\operatorname{th}\alpha_n k+\frac{\pi}{4}\alpha_n J_0^2(\alpha_n)}\cdot\left[\frac{\sin(\alpha_n+\alpha_m)}{\alpha_n+\alpha_m}\right.\\
&\left.+\frac{\sin(\alpha_n-\alpha_m)}{\alpha_n-\alpha_m}(1-\delta_{mn})\right]\quad(m = 1,\ 2,\ 3,\ \ldots),
\end{aligned}\right\} \tag{16}$$

where

$$\gamma_0 = \beta_0\left(1+\frac{4k}{\pi}\right);\quad \gamma_n = \beta_n\left[1+\frac{4}{\pi}\cdot\frac{\operatorname{th}\alpha_n k}{\alpha_n J_0^2\,\alpha_n)}\right];$$

$$\delta_{mn} = \begin{cases}1 & m = n,\\ 0 & m \neq n.\end{cases}$$

It is evident here that the sum of the squares of the diagonal coefficients in the infinite system of algebraic equations satisfy the following inequality:

$$\sum_{m=1}^{\infty}\left|\frac{\operatorname{th}\alpha_m k \sin 2\alpha_m}{\left(\operatorname{th}\alpha_m k + \frac{\pi}{4}\alpha_m J_0^2(\alpha_m)\right)2\alpha_m}\right|^2 \leqslant \frac{1}{4}\sum_{m=1}^{\infty}\left|\frac{\sin 2\alpha_m}{\alpha_m}\right|^2 < \infty,$$

i.e., one can solve system (16) by the reduction method and one can estimate the accuracy of the approximate solution, as long as 1 is not an eigenvalue of the system [5]. The original problem is then solved without difficulty by a successive application of formulas (16) and (11).

The result obtained so far yields only a numerical solution to the problem, however, which is not very useful for a direct analysis. In view of this, it becomes worthwhile to consider an approximate method of determining the temperature field of the system in Fig. 1b which will yield a rather simple analytical solution.

For this purpose, we define the temperature distribution in the $z = 0$, $r \leq 1$ plane in terms of a power series

$$T(0, r)|_{r\leqslant 1} = \chi(r) = \sum_{N=0}^{N} a_m r^m. \tag{17}$$

The expression for $T_2(r, z)$ can then be written as

$$T_2(r, z) = -\frac{z}{k} + 2\frac{z+k}{k}\sum_{m=0}^{N}\frac{a_m}{m+2}$$

$$+ 2\sum_{n=1}^{\infty}\frac{\operatorname{sh}\alpha_n(z+k)}{\operatorname{sh}\alpha_n k}\cdot\frac{J_0(\alpha_n r)}{J_0^2(\alpha_n)}\left[\sum_{m=0}^{N} a_m\int_0^1 \rho^{m+1} J_0(\alpha_n\rho)\,d\rho\right]. \tag{18}$$

The solution for $T_1(r, z)$ is given by formula (9), where $A(\lambda) = B(\lambda)/\lambda$ is found by solving the pair of integral equations

$$\int_0^\infty B(\lambda) J_0(\lambda r)\,d\lambda = \chi(r), \quad r \leqslant 1,$$

$$\int_0^\infty \lambda B(\lambda) J_0(\lambda r)\,d\lambda = 0, \quad r > 1,$$

with the boundary condition for $T_1(r, 0)$ at $r > 0$ taken into account.

The solution to these equations is obtained by the substitution [6]:

$$B(\lambda) = \int_0^1 \varphi(t)\cos\lambda t\,dt, \tag{19}$$

where

$$\varphi(t) = \frac{2}{\pi}\cdot\frac{d}{dt}\int_0^t \frac{r\chi(r)\,dr}{\sqrt{t^2 - r^2}}. \tag{20}$$

For the determination of the unknown coefficients α_m one must use condition (4), which is satisfied at $N + 1$ points selected in some manner. In order to simplify the calculations and improve the accuracy, however, it is worthwhile to determine the coefficients from the condition of equal thermal fluxes impinging on definite segments of the interface between subregions Ω_1 and Ω_2:

$$Q_1(\rho_i) = Q_2(\rho_i) \quad (i = 1, 2, \ldots, N+1), \tag{21}$$

where

$$Q_j(\rho_i) = -2\pi\int_0^{\rho_i}\left.\frac{\partial T_j}{\partial z}\right|_{z=0} r\,dr \quad (j = 1, 2). \tag{22}$$

For instant, for Q_1 one easily obtains the expression

$$Q_1(\rho_i) = 2\pi\left\{\int_0^{\rho_i}\varphi(t)\,dt + \int_{\rho_i}^1\left[1 - \frac{t}{\sqrt{t^2-\rho_i^2}}\right]\varphi(t)\,dt\right\}, \tag{23}$$

TABLE 1. Coefficients a_0 and a_2 for Various Values of Parameter k

k	ρ_i 0,4	0,6	0,8	0,4	0,6	0,8
	a_0			$-a_2$		
0,02	0,9959	0,9988	0 9993	0,0407	0,0465	0,0502
0,06	0,9601	0,9652	0,9791	0,0611	0,0710	0,0983
0,1	0,9338	0,9375	0,9573	0,0903	0,0973	0,1345
0,5	0,6768	0,685	0,688	0,1157	0,1294	0,1349
1,0	0,4921	0,4937	0,4939	0,0885	0,0989	0,099
2,0	0,319	0,320	0,321	0,0575	0,0648	0,0642

derived using the value of the discontinuity integral

$$\int_0^\infty \cos\lambda t J_1(\lambda\rho)\, d\lambda = \frac{1}{\rho} \quad \text{for} \quad t<\rho,$$
$$= \rho\left[\sqrt{t^2-\rho^2}\left(t+\sqrt{t^2-\rho^2}\right)\right]^{-1} \quad \text{for} \quad t>\rho.$$

Expression (23) becomes much simpler for $\rho_i = 1$. With a form of function $\chi(r)$ specified and with the relations in [3] taken into account, it becomes

$$Q_1(1) = 4\int_0^1 \frac{\chi(r)\, rdr}{\sqrt{1-r^2}} = 2\sum_{m=0}^N a_m B\left(\frac{m+2}{2}, \frac{1}{2}\right)$$
$$= 2\sqrt{\pi}\sum_{m=0}^N a_m \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{3}{2}\right)}. \tag{24}$$

We will now retain only the first three terms in expression (17), while obviously $a_1 \equiv 0$ from symmetry considerations,

$$\chi(r) = a_0 + a_2 r^2.$$

Equating pairwise the expressions for the total fluxes per area elements r = 1 and r = ρ_i in the auxiliary problems of internal and external heat transfer based on formulas (22), (23), and (24), we arrive at the following system of equations for the unknowns a_0 and a_2:

$$a_0\left(1+\frac{4k}{\pi}\right) + a_2\left(\frac{1}{2}+\frac{8k}{3\pi}\right) = 1, \tag{25}$$

$$a_0\left[1+\frac{4k}{\pi}\cdot\frac{(1-\sqrt{1-\rho_i^2})}{\rho_i^2}\right] + a_2\left\{\frac{1}{2}+\frac{8k}{3\pi}\left[1 - (1+2\rho_i^2)\sqrt{1-\rho_i^2}\right] + \frac{8k}{\rho_i}\sum_{n=1}^\infty \frac{\operatorname{cth}\alpha_n k J_1(\alpha_n\rho_i)}{\alpha_n^2 J_0(\alpha_n)}\right\} = 1. \tag{26}$$

For k = 0 system (25)-(26) has a unique solution a_0 = 1 and a_2 = 0 at every $\rho_i \in (0.1)$, i.e., we have arrived at the original problem with k = 0.

Knowing the values of the coefficients, it is not easy to determine the temperature distribution within the entire half-space z ≥ 0. Thus, for the temperature field along the disc axis we have

$$T(0,\ z) \approx T_1|_{r=0} = \frac{2}{\pi}\left[(a_0 - 2a_2 z^2)\operatorname{arctg}\frac{1}{z} + 2a_2 z\right], \tag{27}$$

and at z = 0

$$T(r,\ 0) \approx T_1|_{\substack{z=0\\ r>1}} = \frac{2}{\pi}\left[(a_0 + a_2 r^2)\arcsin\frac{1}{r} - a_2\sqrt{r^2-1}\right]. \tag{28}$$

It is to be noted that, by finding the coefficients a_m (for instance, a_0 and a_2 from Eqs. (25)-(26)) at various values of ρ_i, one can, at the same time, indirectly estimate the accuracy of the obtained approximate solution for the mathematical model in Fig. 1b without comparing it with the results based on exact formulas or with the results of temperature field simulation. Some results of these calculations for various values of parameter k are given in Table 1.

We note, in conclusion that the potential distribution across a disc plane at k > 1 may, with great accuracy, be assumed uniform. In this case the expression for the temperature becomes

$$T \approx \frac{2}{\pi + 4k} \arcsin\left(\frac{2}{\sqrt{(1-r)^2+z^2}+\sqrt{(1+r)^2+z^2}}\right). \qquad (29)$$

The same expression can be easily obtained from the solution to system (15), if only the first term is sought and all other terms are disregarded, i.e.,

$$\beta_0 = -\frac{2}{\pi} \cdot \frac{1}{1+\frac{4k}{\pi}}.$$

NOTATION

T	is the temperature;
Bi	is the Biot number;
Q_j	is the total thermal flux;
J_0, J_1	are the Bessel functions, of the zeroth and of the first order;
$F(\alpha, \beta, \gamma, z)$	is the hypergeometric function;
B(x, y)	is the beta function;
Γ (z)	is the gamma function;
f, g, A, B, χ, φ	are the function symbols;
r, z	are the cylindrical coordinates;
λ, ρ, t	are the variables;
β_n, γ_n, a_m	are the unknown coefficients.

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